Mathematics as a Source of Certainty and Uncertainty

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Ladies and gentlemen,

It is difficult for me to express adequately my appreciation of the honour which has been conferred upon me by inviting me to address this audience. So I immediately start with the topic of this lecture, but before I have to stress that its genre is unusual for me. Unlike in lectures any mathematician is used to deliver, I'm not going to try to increase your store of knowledge. I'll rather express some feelings and a certain mood. For some technical reasons the sources I could use in preparing this lecture were scarce. This explains the excessive use of my personal experience and memory for which I apologize in advance.

"Certainty" seems to be the most appropriate word to express what the non mathematical public thinks of mathematics. This opinion could be supported by quotations of famous mathematicians, though there is no lack of scepticism in what mathematicians think and say about their subject. But the non mathematical public has no doubt that mathematics is a realm of certainty. This feeling is well expressed by the following saying of the French painter Georges Braque: "Art upsets, science reassures" (if we replace the word "science" by "mathematics").

This popular (and completely mistaken) opinion determined my destiny. It is thanks to it that I became a mathematician. The decision was actually taken not by me, but by my father, a philologist whom I fully obeyed at the time, though my mathematical achievements had been nothing more than good highschool marks. According to my natural inclinations I'd rather become a philologist too. But I graduated from the highschool in 1950, the year when Stalin also got interested in philology and wrote a booklet "Marxism and problems of linguistics". This event, memorable for everybody of my generation had been preceded by several ideological campaigns when politicians taught (sometimes with the use of violence) writers, historians, musicians, and literary critics what is good or bad. Impressed by this practice, my father forbade me even to think about the humanities and *ordered* me to become a mathematician. Now, 43 years later I'm grateful to him for this risky decision. But he was completely mistaken in what concerns his main point, namely, search for certainty. Of course, politicians hardly can prescribe to mathematicians which theorems to prove. Nevertheless, mathematics is the worst place to look for certainty and definiteness. It is, to the contrary, a realm of uncertainty. This uncertainty, its malignant and beneficial sides, are the subject of this lecture.

I'll describe how one acquires and then loses the comforting feeling of certainty usually ascribed to mathematics. Then I turn to a kind of uncertainty which I consider as beneficial, and which is supported by mathematical thought in an essential way.

My first impressions from mathematics fully agreed with the opinion of Braque. This science reassured indeed. I was fascinated by the preciseness and expressive power of the language. Mathematicians are much more delicate, cautious, and I'd say, nervous word users than anybody else. Unlike others, they are really bothered with the *meaning* of words they are pronouncing or writing down. I was deeply impressed by the capacity of mathematical language to describe – in an unequivocal and very expressive way – and fix very numerous and heterogeneous ideas, ranging from analysis to probability, from classical to quantum mechanics, from economics to linguistics.

The convincing power of mathematical proofs seemed overwhelming, irresistible to me, exceeding by far any proof in any domain where verbal proofs play an essential role, be it physics, history, or law.

There is one more component of this special certainty insidiously conquering any young mathematician as he gradually opens up his profession. It is something spontaneously felt by any mathematician in spite of the fact that some would deny it and notwithstanding well founded criticism of logicians and philosophers. It is a deeply rooted, visceral belief in (or rather a sensation of) the existence of a special mathematical reality "hard as rock" according to Hardy who praised this transcendental world in his "Mathematician's Apology" claiming that the mathematical reality is more real than the physical one. It is a belief making an active mathematician insensible to remarkable results of logic which warn him against this "naive superstition". He is just unable to question the existence of things whose properties it is his task to conceive and whose very real resistance depriving him of sleep and rest he permanently tries to overcome. He reacts to theorems of Gödel and Cohen, to critical attitudes of intuitivists and constructivists with a mixture of respect and vague feelings of guilt, forgetting all this in the everyday communication with the stubbornly existing mathematical reality. As to me, the loss of certainty came not from logic. Its origin was of lower, much more earthly and practical level. My first doubts can be squeezed to quite silly, childish questions: "What for?" "What is the aims of mathematics?" "What is good and what is bad in it?" "What are the criteria of value?" Now I know that these questions just cannot be answered. But then, in the fifties, I have been really upset, after it became clear to me that nobody can provide me with a satisfactory and comforting answer. Eventually I had to accept this situation and to live on. Now I can say that mathematics, being a beautiful, miraculous science, is, at the same time, subject to fashion and cult of power. Its value criteria (at least those applied in practice) are very often determined by market forces and whimsical, arbitrary and irrational opinions and tastes. I could illustrate this sad assertion by several funny stories and almost every mathematician could add his. Let me only briefly mention a curious fate of the Cantor set which I choose as a symbol of a domain inhabited by species usually called "bad sets" or "bad functions". For the generation of my teachers they symbolized the progress. I was brought up in deep respect of these objects and related ideas and techniques. But soon after I've graduated from the university I knew that these favourite things of my teachers had become obsolete, a mark of backwardness. It became fashionable to say that "bad functions do not exist". The term "Theory of functions of a real variable" acquired an abusive nuance. At that time I often heard from my colleagues that the attention paid in the twenties and thirties to bad sets and functions in Russia and Poland was a kind of decadence and degeneration distracting mathematics from its true destination which is to solve problems of physical origin. But what do we see now? The Cantor set is fashionable again! Masses of people are really obsessed by it (and similar objects) claiming that physics (physics!) just would perish without them. Luxurious volumes of pictures are being printed and successfully sold, the Cantor set and its relatives got a new name, they are "fractals" now; they are not "bad", but "beautiful" (everybody knows the title "The beauty of fractals"). Of course, this boom is related to really deep discoveries in the theory of dynamical systems, new understanding of chaos. Grimaces of vanity and fashion, market tendencies in mathematics coexist with significant development of thought, only masking and distorting it. The above description does not contradict the well-known metaphor comparing mathematics with an orchestra whose participants don't know each other being separated by distances and interdisciplinary barriers, but the orchestra is nevertheless perfectly concerted, producing divine music, as if it were led by an invisible Conductor. This *is* true, but this image can be perceived only from afar, and nobody has ever seen the score. The whereabouts of the Conductor and his plans are obscure, and nobody, no group, no organization can claim his role.

But in the real life a mathematician is often in a situation where he has to judge, to accept or reject. Those who are obliged to accept or reject papers for publication or applications for a job deserve to be pitied. The total lack of formal, algorithmic criteria of selection makes their situation highly unpleasant. If your department got 500 applications, then usually it is not hard to reject 450 according to reasonable and sound considerations. But what if you have only 2 positions, and the rest of applicants consists of good, serious specialists but does not contain, say, a Gauss and a Hilbert? Then you make a clever face and say that a class of spaces, the favourite theme of candidate X, is not worth considering, or that the theorem of Y is good but not a breakthrough, and a theorem of Z *is* a breakthrough, but the number of complex variables is one, and this is old-fashioned.

I'm not criticizing. I have no proposals. I'm describing. In such situations there is no way to escape subjective conclusions conforming with personal tastes. The only thing which could be avoided is to pretend that you possess the objective truth and are motivated by superscientific considerations.

I remember a lecture delivered by P. Aleksandrov, the famous topologist, in Leningrad, somewhere at the end of the sixties. Its title was "Criteria of value in mathematics". He analyzed, one after another, three criteria: applicability, fashion, degree of difficulty – only to reject them all. He proposed instead something very indefinite like "a feeling of a new horizon". This is by far not an algorithmic solution. But I prefer it to the terrible practice when the works of a person are judged according to the journal which published them. This is very algorithmic indeed and frees you from reading mathematics which requires concentration, energy and time.

So far about the criteria of value in mathematics and understanding its own aims and necessity. It is uncertainty in its purest and very unpleasant form.

But, to conclude with an optimistic note, let us turn to a beneficial kind of uncertainty inherent in mathematics and contrasting its malignant forms.

Mathematics is often and deservedly lauded for its applications to other sciences. It is impossible to deny these merits of mathematics whose very existence always was determined (and still is) by a subtle interplay of exterior and interior incentives. But I want to emphasize not the applications, but a capability to create sound and reasonable *doubts* and *uncertainty*, things which are in a short supply, but very necessary nowadays. In this connection we could remember again great achievements of logicians, but I'll dwell on much more elementary, almost highschool matters.

Any mathematician, unlike (unfortunately) other people, knows (not only knows, but has it in his flesh and bones) that not everything which has got a name exists in reality. Mathematicians are *professionally* obsessed by existence problems. And not only by the existence of an *object* (a solution, a function or a set with prescribed properties), but by existence of a solving procedure when the algorithm in question has to satisfy certain requirements.

Normal people, not trained in this school of professional doubt, confronted with any problem, rarely suspect it can be unsolvable. They just start solving it. This is normal. And this is awful. Consider the following series of isomorphic proposals.

Let us trisect an angle using compass and ruler only, let us construct a perpetuum mobile, or "socialism"; let us do away with inflation and unemployment. Mathematics contains a powerful sobering potential suggesting how cautiously you must react to these appeals. Creating and propagating reasonable uncertainty and doubt, mathematics is capable to calm down, to cool away many dangerous and contagious enthusiasms.

In the sixties it became fashionable to jeer and sneer about compass and ruler problems in the highschool ("Why compass and ruler, why not something else?"). The jokers seem to be the same people who produced awful highschool geometry books with axioms of a vector space preceding triangle and circle. There are several strong arguments in defence of the compass-ruler problems, but I emphasize only one: it so happened that just these ancient problems served as the material for discoveries whose contribution to culture is tremendous and whose results need to be inculcated into the mass psyche, to become a commonplace: NOT EVERY PROBLEM CAN BE SOLVED. This is the main reason to include these problems into the highschool teaching. They can be explained to every schoolboy and schoolgirl producing a salutary pedagogical influence. Being acquainted with the procedure of bisection of an angle, it is natural to start thinking about trisection. Why not? These problems are so similar! The non-solvability of the second is highly not obvious. Nevertheless it *is* unsolvable and this can be rigorously *proved.* Mathematics abounds in results of this kind when something seems to be within one's reach but eventually turns out to be impossible. But denying the possibility to find or do something, mathematics yields some consolations in the form of *approximate* solutions, *optimization* algorithms suggesting the ideology of *compromise.* A person brought up in this spirit hardly can join a crowd crying like mad "liberté, égalité, fraternité" only to start mass killings afterwards. An easy reasoning will lead this person to the conclusion that the terms of this triad are not compatible with each other, and it is better to look for something approximate, but feasible.

For ages the general human aspiration was to catch and freeze everything as *notions*, creating all-embracing and all-explaining systems of thought. Isn't it clear now that this is only possible with relatively trivial things? The real complexity of world can be only *approximately* described, and this description cannot manage with *notions*, it needs *images*. Mathematics is a source of a lot of images, not less expressive than images of poetry. Penetrating your heart they are capable to influence your world perception.

Let me use two quotations, one due to a famous sociologist, and another to a humorist. "Many of the greatest things man has achieved are not the result of consciously directed thought, and still less the product of a deliberately coordinated effort of many individuals, but of a process in which the individual plays a part which he can never fully understand" (von Hayek). The second quotation is much shorter: "No snowflake in an avalanche feels responsible" (Jerzy Lec). But in my feeling, vanity and *futility* of individual efforts hardly can be expressed with a greater force than by the following "uncertainty theorem": the value of the Lebesgue integral does not depend on values of the integrand on any prescribed set of zero length.

The theorem is a flagrant expression of senselessness of such notions as "cause", "guilt" or "responsibility" applied to results of sufficiently massive, integral character. Meanwhile every Russian traveller is being asked daily: "What do you think about Gorbachev or Yeltsin" as if these men (or anybody else) can be considered as causing or governing immense, cosmic changes going on in Russia. Returning to the "damned questions"* of mathematics ("what for?" "what is good or bad?") we can again use the above theorem as an instructive image. Results of work of the mathematical community at any moment before the 2nd World War could be expressed as a SUM of individual efforts. By the end of the sixties the set of terms of this sum became practically infinite (though, probably, still countable). But now this sum definitely has become an integral. I think this is a *Lebesgue* integral of personal efforts, though the presence of an infinite set of point masses may be arguable. I'd rather admit a singular *continuous* component with no distinguishable separate points. But let us agree, at least, that this image has a right to exist.

It can cool the incandescence of passions and weaken prohibitive trends. Nobody can or must feel or claim responsibility of mathematics as a whole. Its sense, its message are as inconceivable as life. It is an integral, and preoccupations inspired by fashion are vanitas vanitatum.

Powerful images carrying mighty expressive charge are connected with analytic functions and their antipodes, so called "bad" functions. Creators of Analysis were spontaneously convinced that all functions are analytic (even before this term had been coined). This spirit has been weakened as a result of the "string dispute" at the end of the XVIII century. But in a milder form this frame of mind generally persisted even in XIXth century.

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* Literal translation of a Russian expression denoting the most fundamental problems concerning man's essence and existence.

Nature needs analytic functions only. Such was the healthy, elemental view of people believing in God, predeterminacy and predictability of Being. A property drastically opposing an analytic function to a "bad" one is *uniqueness*.

Past and future of a process described by an analytic function are completely determined by its course during a second, or a one millionth part of a second. This mathematical image is apt to create mixed emotions. An analytic function is a symbol of highest perfection as is a favourite melody or line of poetry. Starting first notes or words it is impossible to continue differently from the classical sample. But at the same time the analyticity is a severe verdict, an inflexible prediction, impossible to contest. If an analytic curve $y = f(t)$ coincides with the parabola $y = t^2$ on (0,1), then these two curves are doomed to coincide forever, no choice is possible. There is something very significant in the reluctance of old classics to accept the possibility to represent "an arbitrary" curve as a sum of trigonometrical series, such representation being "a formula". But all functions defined by formulas have to be analytic and cannot change their course in an arbitrary way.

Oscillations of fashion around "bad" and "good" functions mentioned above reproduce, in a sense, the old "string dispute". In spite of its vagueness, abundance of terms not duly defined, absurd claims and personal biases, this dispute includes something really important. Human beings can be divided into two categories. The first one believes (or feels) that world is described by analytic functions. For the second everything is expressed by Lebesgue measurable functions. At any moment their course is absolutely unpredictable and, hence, can (in principle) be changed in any desired way. So, this second attitude implies *certainty*, those people feel they are masters of world. Of course, no argument is thinkable here. We are dealing not with clear statements to be proved or disproved. We are dealing with different psychological approaches to reality, with conflicting world orientations. I dare to express my deeply personal, non-verifiable, non-arguable confidence in the analyticity of the world. Chaotic behaviour results from the interaction of innumerable analytic processes. This irrational feeling is warranted by some rigorously proved mathematical facts. However wild a function of time might look, it is, eventually, the sum of a series of polynomials or a difference of two analytic functions.

I used these elementary examples to show how some images so familiar to any mathematician can suggest the noble habit of doubt and strengthen the feeling of beneficial uncertainty.

This eulogy of uncertainty and doubt I'm finishing to deliver is not something unusual nowadays. The mood I tried to express is gaining more and more room, undermining certainties and selfconfidence of conceited leaders, making it harder to politicians to subdue masses by cheap incantations devoid of any real content. After I've already sent the title of this lecture to professor Hedberg, in a Montreal airport I bought "Le Monde" of the 21st of April with an article of Edgar Morin, French Socialist, "La pensée socialiste en ruine", and was surprised to read the following lines (a newspaper is the last place where I could dream of finding something useful for this lecture): "In the opinion of Marx science is a *source of certainty*.

But today we know that sciences yield *local certainties*, but theories are scientific insofar as they can be refuted, that is, are *not certain*. And, in what concerns fundamental questions, the scientific cognition runs into bottomless *uncertainties*. For Marx, the scientific *certainty* eliminated philosophical interrogation. Today we see that scientific progress only animates fundamental philosophical problems."

The attitudes I expressed become more and more banal which is illustrated by their frequent appearance even in the mass media. The more banal, the more commonplace they become, the more is our hope for the eventual improvement of the world, more human relations between human beings. And I hope that the experience accumulated in mathematics, joint with the experience of everyday practice, history, philosophy, positive sciences, religion and art will contribute to making these attitudes of beneficial uncertainty a commonplace indeed.