IN MEMORY OF GENNADII M. GOLUZIN (1906–1952)

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Gennadii Mikhailovich Goluzin was born in 1906 in the old Russian town of Torzhok to a worker's family. In 1924, he entered the department of mathematics and mechanics of the Leningrad State University. In the beginning of 1929, he defended his diploma work, which was published in the same year in *Matematicheskii Sbornik*. At that time, his teaching activity began. He defended his doctoral dissertation brilliantly in 1936. From 1938, Goluzin headed the department of the theory of functions of a complex variable at the Leningrad State University. From the moment of foundation of the Leningrad Branch of the Mathematical Institute of the Academy of Sciences of the USSR (now, the St. Petersburg Branch of the V. A. Steklov Mathematical Institute of the Russian Academy of Sciences) and till the end of his life, Goluzin also worked at the institute.

The beginning of Goluzin's scientific activity dates back to the 30s. His first papers were devoted to problems of mathematical physics. Here it is appropriate to mention the familiar Carleman–Goluzin–Krylov inequality. However, already in the mid-30s, Goluzin had turned to geometric function theory. It should be mentioned that in the 20s and 30s this area still was in the making. The first elementary method of geometric function theory was the area method based on the principle of nonegativity of area. In 1916, L. Bieberbach used the method to establish various remarkable properties of the Koebe function $K_\varepsilon(z) = z(1-\varepsilon z)^{-2}$, $|\varepsilon| = 1$, and conjectured that the inequality $|c_n| \le n$ holds true in the class S of the functions that are regular and univalent in the disk $|z| < 1$ and have the expansion of the form

$$f(z) = z + c_2 z^2 + c_3 z^3 + \dots$$

in the vicinity of the origin, and $|c_n| = n$ only for the Koebe function. Bieberbach himself proved the inequality $|c_2| \le 2$.

Loewner's parametric method was the first deep method of geometric function theory, and Goluzin played a crucial role in developing and popularizing this method. It is known that the aim of C. Loewner's paper of 1923 was to prove an estimate of the modulus of the third coefficient in the class S. Only a few years after the publication of Loewner's paper, Goluzin turned to the parametric method and used it for the uniform derivation of the main results of the theory of univalent functions. At that time, Goluzin used this method to obtain new results, including the precise form of the rotation theorem, i.e., the sharp estimate of $|\arg f'(z)|$ in the class S for $|z| = r$, $0 < r < 1$. Goluzin repeatedly returned to the Loewner method in succeeding years. At present, the Loewner method is one of the main methods of geometric function theory.

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In 1965, C. Loewner participated in the International Conference in Yerevan (Armenia). When speaking with Leningrad mathematicians, he said that he had always marveled at the profound evolution of his old work of 1923.

Grötzsch developed his strip method at the end of the 20s and the beginning of the 30s. The method is based on relations between length and area; it considers characteristic conformal invariants of doubly connected domains and quadrangles. Using this method, Grötzsch obtained a large number of deep results for simply connected and multiply connected domains. However, Grötzsch's papers did not gain proper recognition for a long time; one of the reasons for this could be the isolation of German scientists of that time from the rest of the scientific community. Goluzin was one of the first to appreciate the possibilities of the method: he obtained various applications of the strip method in a series of papers in the 30s. Using this method, Goluzin for the first time proved the theorem saying that the image of the unit disk under the mapping by a function from the class S contains n segments emanating from the origin at equal angles and such that the sum of their lengths is at least n. Afterward, Grötzsch's strip method provided the basis of the method of the extremal metric, which is widely used at present in geometric function theory and has found applications in other areas of mathematics.

The calculus of variations for univalent functions differs substantially from the classical calculus of variations, since the classes of univalent functions are nonlinear in a high degree.

Schiffer created the method of boundary variations in 1938 and the method of inner variations in 1943. The first applications obtained by Schiffer were mainly qualitative results for extremal functions in the problem on the maximum of the modulus of coefficients in the class S. Goluzin developed his version of the method of inner variations in a series of papers in 1946–1951. Goluzin's proof, elementary in the main, is based on the properties of majorant power series. Goluzin applied his method of variations to various problems of the theory of univalent functions. The results obtained by Schiffer's and Goluzin's variational methods show the essential role of quadratic differentials in solutions of extremum problems. In a number of cases, the proofs obtained by Goluzin's variational method turn out to be much simpler than those based on Schiffer's method. Using his method, Goluzin obtained various results in the Chebotarev problem and in the problem on the maximum of the nth diameter (these questions play an important part in the theory of capacity of planar sets), in the problem of the maximum of the product of powers of conformal radii of nonoverlapping domains, and in various distortion theorems. Among them is the familiar inequality for the values of the function $f(z) \in \Sigma$ at a collection of points from the domain $|z| > 1$ (here, Σ is the class of the functions $f(z)$ that are meromorphic and univalent in the domain $|z| > 1$ and have the Laurent expansion of the form $f(z) = z + \alpha_0 + \alpha_1 z^{-1} + \ldots$ in the vicinity of $z = \infty$):

$$\left| \sum_{\mu,\nu=1}^{n} \gamma_\mu \gamma_\nu \log \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \right| \leq \sum_{\mu,\nu=1}^{n} \gamma_\mu \overline{\gamma}_\nu \log(1 - z_\mu^{-1} \overline{z}_\nu^{\,-1}).$$

Goluzin's scientific heritage is very diverse and far from being exhausted by the results cited above. To give only a few examples, let us mention his paper on p-valent functions, an extensive study of inner properties of functions from the Hardy classes, and results on univalent functions in multiply connected domains. Goluzin's results initiated various directions of study in geometric function theory and made a strong impact on the modern problematics of this theory.

Goluzin devoted considerable attention to propagating the ideas and results of geometric function theory. His brilliant erudition and facility in presentation allowed him to write several bright surveys on geometric function theory. One of them was the extensive survey "Inner problems of the theory of univalent functions," published in 1939 in *Uspekhi Matematicheskikh Nauk*. This paper was one of the first surveys devoted to geometric function theory in the world mathematical literature. The continuation of the survey was published in 1949 in a separate volume of *Trudy MIAN*.

Several generations of mathematicians learned from Goluzin's monograph *Geometric Theory of Functions of a Complex Variable*. The first edition of this book was published in 1952, and the second enlarged edition in 1966; the book was translated into German and English. The monograph is an encyclopedia by its contents: in addition to the methods of geometric function theory (Loewner's parametric method, method of variations, Grötzsch's strip method, and other methods), it handles general questions of the theory of conformal mapping of simply connected and multiply connected domains, metrical properties of closed sets, various majoration principles, and boundary properties of analytical functions. This monograph is a standard reference book of modern analysts.

Many mathematicians, including one of the editors of the present volume, first became acquainted with Goluzin as the editor of an excellent translation of the two-volume *Course in Modern Analysis* by E. T. Whittaker and G. N. Watson, published in 1934.

Goluzin gave much consideration to teaching. In addition to the basic course in the theory of functions of a complex variable in the faculty of mathematics and mechanics of the Leningrad State University, Goluzin read various special courses and led two seminars in geometric function theory: a seminar for undergraduate students and a more advanced seminar. The author of the present note belonged to the last group of students of the faculty who were specializing in geometric function theory under the supervision of Goluzin. I remember how Goluzin asked me to present a talk in the undergraduate seminar about the proof of the Lavrent'ev–Shepelev–Rengel theorem on $\sqrt[4]{\frac{1}{4}}$ obtained by the strip method, which was given in his survey article of 1939 (later on, the proof was included in his monograph). Some points of the proof were unclear to me at first. Goluzin at once understood the drawing that I showed him. "Your drawing is too complicated," he said. "These pieces just should not be considered at all." (He meant the parts of the domain that were cut off by rectilinear segments.) The geometric proof of the theorem on $\sqrt[4]{\frac{1}{4}}$ suggested by Goluzin is a bright illustration of the basic ideas of Grötzsch's strip method. Afterward, it served as an example helping me to understand the geometrical meaning of a number of proofs based on the method of the extremal metric.

Goluzin was very considerate to his students. I remember very well how in the beginning of the summer of 1951, before student vacations, Goluzin, already seriously ill, invited us to his place and gave us the subjects of our diploma works in good time. "You will work in the summer, won't you?" Goluzin asked us at parting, obviously hoping to hear "yes."

Many well-known mathematicians had "grown up" in Goluzin's seminar on geometric function theory. Among Goluzin's students were Yu. E. Alenitsyn, S. A. Gel'fer,

L. I. Kolbina, N. A. Lebedev, Yu. D. Maksimov, I. M. Milin, and L. N. Slobodetskii. Many of Goluzin's investigations were continued in the works N. A. Lebedev and I. M. Milin; as for Goluzin, geometric function theory became the main work of their lives.

After Goluzin, Lebedev led the seminar for thirty years (1952–1982). The work of the seminar followed the traditions laid by Goluzin. In those years, the Leningrad seminar attracted numerous participants, and many mathematicians from other cities presented talks in the seminar.

The seminar continues his work as before. In 1984, the Leningrad seminar in geometric function theory listened to Louis de Branges's talks devoted to his proof of the Bieberbach conjecture. It is worth noting that a substantial role in the proof is played by Loewner's method, and the proof uses a coefficient inequality obtained by Lebedev and Milin.

Goluzin was a very modest, kind, and unassuming person in everyday life. Working at LOMI, he performed his duties with a great degree of responsibility. Even after receiving the State Prize of 1947, and also the Leningrad University Prize of 1946, he was still very much concerned with annual scientific reports, which he was to present as a research fellow of LOMI.

Goluzin had to live and work under difficult conditions. He and his family survived the extremely hard blockade winter in Leningrad. The meal rations alloted by the Government to scientists of the Leningrad University were a substantial and probably deciding help. The Steklov Mathematical Institute was evacuated to Kazan' for the duration of the war. After returning from Kazan' in 1944, Goluzin, together with his wife, Antonina M. Chufistova, and their three daughters, Lena, Anya, and Masha, continued to live in a room of a large communal flat on the fourth floor of an old six-storied Leningrad house (on the corner of the Liteinyi prospect and Nekrasov street).

When a student, I often saw Goluzin walking with his children. His daughters recall how he taught them to play chess and also other games invented by him, and how he brought them to the then well-known shop at the beginning of the Nevskii Prospect, which used to have a rich choice of geographical maps, globes, and other visual aids in geography. Goluzin's favorite kind of relaxation was amateur photography. His unrealized dream was to undertake a long journey. All three of Goluzin's daughters became mathematicians.

Goluzin's devotion to science was extraordinary. He continued his work enthusiastically in what seemed to be impossible conditions. I. M. Milin, who visited Goluzin during the latter's last days, used to recall that Goluzin greeted him with the question: "Have you proved anything of interest?"

Goluzin's role in the development of geometric function theory in our country can hardly be overestimated. His contribution to function theory has been highly regarded by the international mathematical community. Monographs of Russian and foreign mathematicians devoted to geometric function theory contain numerous references to Goluzin's papers.

Translated by N. Yu. Netsvetaev.